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Efficiency comparison of split-block design versus split-plot design*

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SUMMARY

Introducing an additional column structure in a split-plot design leads to a split-block design where the superblocks have a lattice structure. This paper investigates the relative efficiency between this two types of designs including the incomplete case.

KEY WORDS: nested row-column structure, split-plot designs, split-block designs, relative efficiency.

1. Introduction

In many experiments one has the option of choosing from a wide class of designs for the purpose of precise experimental results. A measure of the efficiency of one design relative to another is the inverse ratio of the variances with which treatment differences are estimated in the two designs, i.e.:

$$RE(\text{design A versus design B}) = \frac{\text{efficiency A}}{\text{efficiency B}} = \frac{\text{var B}}{\text{var A}}.$$

Given that design A has been used, estimation of its efficiency relative to design B requires that we estimate the error variance of design B from the data obtained from design A.

The efficiency problem of experimental designs has been explored by many researchers (e.g., Yates, 1935, Cochran & Cox, 1957, Pearce, 1983, Lentner et al., 1989; etc). Many well known designs have already been accounted for in monographs, with

*The paper was submitted on the occasion of 70-th birthday of Professor Tadeusz Caliński.

some new designs remaining uncovered. One of them is the split-block design or split-plot design in stripes (Hering and Mejza, 1997). Suppose that a split-block design has been used, then the experimenter may ask: what would have been the error variance if a split-plot design had been used? Or to put it in another way, what is the efficiency of a split-block design relative to a split-plot design?

2. The linear models

A split-block design can be obtained from a split-plot design by adding an additional column structure to the superblocks. So the basic difference between the split-plot design and the split-block design is the structure in which the levels of the two treatment factors are assigned to the experimental units (Hinkelmann and Kempthorne, 1994). In a split-block design, both factor A with levels A_1, $A_2, ..., A_s$ and factor B with levels B_1, $B_2, ..., B_t$ are applied to whole-plots which are "orthogonal" to each other. A model reflecting this structure is of the form

$$y_{ijk} = \mu + r_i + \alpha_j + e_{ij}^A + \beta_k + e_{ik}^B + (\alpha\beta)_{jk} + e_{ijk}^{AB}, \tag{1}$$

with $i = 1, 2, ..., r$; $j = 1, 2, ..., s$; $k = 1, 2, ..., t$. Here μ is the general mean, r_i is the superblock effect, and α_j, β_k are the row, column effect, respectively, whereas $(\alpha\beta)_{jk}$ is the measure of interaction between rows and columns. e_{ij}^A, e_{ik}^B, and e_{ijk}^{AB} can be considered as i.i.d. random variables with means 0 and variances σ_{eA}^2, σ_{eB}^2 and σ_{eAB}^2, respectively.

If the same experimental material were used as a split-plot design, i.e., a design in which the effects of factor A, occurring at s levels, were measured by the overall response on the whole-plot composed of t split-plots associating to different levels of factor B, then the model would have been

$$y_{ijk} = \mu + r_i + \alpha_j + e_{ij}^A + \beta_k + (\alpha\beta)_{jk} + e_{ijk}^{AB} \tag{2}$$

with $i = 1, 2, ..., r$; $j = 1, 2, ..., s$; $k = 1, 2, ..., t$. Here μ is the general mean, r_i is the superblock effect, and α_j, β_k are the whole-plot, split-plot effect, respectively, whereas $(\alpha\beta)_{jk}$ is the measure of interaction between the whole-plot and the split-plot.

3. Comparing efficiencies when both designs are complete

The ANOVA for the complete split–block design and the complete split–plot design are given in Table 1 and Table 2, respectively.

Following the methods by Yates (1935), using a uniformity trial for both situations, i.e., pooling treatment sums of squares with appropriate error sums of squares,

Table 1. ANOVA for split–block design

Source	d.f.	SS	MS	$E(MS)$
Replicates	$r-1$	$st\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2$	$MS(R)$	
A-factor	$s-1$	$rt\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2$	$MS(A)$	$\sigma^2_{eAB}+t\sigma^2_{eA}+\frac{rt\sum_j\alpha_j^2}{s-1}$
Error(A)	$(r-1)(s-1)$	$t\sum_{i,j}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	$MS(E_A)$	$\sigma^2_{eAB}+t\sigma^2_{eA}$
B-factor	$t-1$	$rs\sum_k(\bar{y}_{..k}-\bar{y}_{...})^2$	$MS(B)$	$\sigma^2_{eAB}+s\sigma^2_{eB}+\frac{rs\sum_k\beta_k^2}{t-1}$
Error(B)	$(r-1)(t-1)$	$s\sum_{i,k}(\bar{y}_{i.k}-\bar{y}_{i..}-\bar{y}_{..k}+\bar{y}_{...})^2$	$MS(E_B)$	$\sigma^2_{eAB}+s\sigma^2_{eB}$
A×B	$(s-1)(t-1)$	$r\sum_{j,k}(\bar{y}_{.jk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2$	$MS(A\times B)$	$\sigma^2_{eAB}+\frac{r\sum_{jk}(\alpha\beta)^2_{jk}}{(s-1)(t-1)}$
Error(AB)	$(r-1)(s-1)$ $(t-1)$	$\sum_{i,j,k}(\bar{y}_{ijk}-\bar{y}_{ij.}-\bar{y}_{i.k}-\bar{y}_{.jk}$ $+\bar{y}_{i..}+\bar{y}_{.j.}+\bar{y}_{..k}-\bar{y}_{...})^2$	$MS(E_{AB})$	σ^2_{eAB}
Total	$rst-1$	$\sum_{i,j,k}(y_{ijk}-\bar{y}_{...})^2$		

Table 2. ANOVA for split–plot design

Source	d.f.	SS	MS	$E(MS)$
Replicates	$r-1$	$st\sum_i(\bar{y}_{i..}-\bar{y}_{...})^2$	$MS(R)$	
A-factor	$s-1$	$rt\sum_j(\bar{y}_{.j.}-\bar{y}_{...})^2$	$MS(A)$	$\sigma^2_{eB}+t\sigma^2_{eA}+\frac{rt\sum_j\alpha_j^2}{s-1}$
Error(A)	$(r-1)(s-1)$	$t\sum_{i,j}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$	$MS(E_A)$	$\sigma^2_{eB}+t\sigma^2_{eA}$
B-factor	$t-1$	$rs\sum_k(\bar{y}_{..k}-\bar{y}_{...})^2$	$MS(B)$	$\sigma^2_{eB}+\frac{rs\sum_k\beta_k^2}{t-1}$
A×B	$(s-1)(t-1)$	$r\sum_{j,k}(\bar{y}_{.jk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2$	$MS(A\times B)$	$\sigma^2_{eB}+\frac{r\sum_{jk}(\alpha\beta)^2_{jk}}{(s-1)(t-1)}$
Error(B)	$(r-1)s(t-1)$	$\sum_{i,j,k}(\bar{y}_{ijk}-\bar{y}_{ij.}-\bar{y}_{i.k}-\bar{y}_{...})^2$	$MS(E_B)$	σ^2_{eB}
Total	$rst-1$	$\sum_{i,j,k}(y_{ijk}-\bar{y}_{...})^2$		

we see from Table 3, that

$$\begin{aligned}&r(s-1)MS_{sb}(E_A)+r(t-1)MS_{sb}(E_B)+r(s-1)(t-1)MS_{sb}(E_{AB})=\\&=r(s-1)MS_{sp}(E_A)+rs(t-1)MS_{sp}(E_B).\end{aligned}\tag{3}$$

Obviously,

$$r(s-1)MS_{sb}(E_A)=r(s-1)MS_{sp}(E_A).$$

Table 3. ANOVA for uniformity trial

(a) Split-block design

Sources	d.f.	MS
Replicate	$r-1$	
Error (A)	$r(s-1)$	$MS_{sb}(E_A)$
Error (B)	$r(t-1)$	$MS_{sb}(E_B)$
Error (AB)	$r(s-1)(t-1)$	$MS_{sb}(E_{AB})$
Total	$rst-1$	

(b) Split-plot design

Sources	d.f.	MS
Replicates	$r-1$	
Error (A)	$r(s-1)$	$MS_{sp}(E_A)$
Error (B)	$rs(t-1)$	$MS_{sp}(E_B)$
Total	$rst-1$	

Thus, (3) reduces to

$$r(t-1)MS_{sb}(E_B) + r(s-1)(t-1)MS_{sb}(E_{AB}) = rs(t-1)MS_{sp}(E_B), \tag{4}$$

and hence

$$MS_{sp}(E_B) = \frac{r(t-1)MS_{sb}(E_B) + r(s-1)(t-1)MS_{sb}(E_{AB})}{rs(t-1)}. \tag{5}$$

The information on the whole-plot treatment comparison in a split-plot design would have been proportional to $1/MS_{sp}(E_A)$ and on the split-plot treatment comparison and interaction proportional to $1/MS_{sp}(E_B)$, whereas in the split-block design information on the row treatment comparison is proportional to $1/MS_{sb}(E_A)$, on column treatment – to $1/MS_{sb}(E_B)$, and on interaction – to $1/MS_{sb}(E_{AB})$. From (5) we know that $MS_{sp}(E_B)$ is a weighted average of $MS_{sb}(E_B)$ and $MS_{sb}(E_{AB})$. Usually, $MS_{sb}(E_B)$ is greater than $MS_{sb}(E_{AB})$ (except for sampling errors), $MS_{sp}(E_B)$ will be intermediate in size between $MS_{sb}(E_B)$ and $MS_{sb}(E_{AB})$.

Because $MS_{sp}(E_A) = MS_{sb}(E_A)$, for comparison of factor A we have

$$ERE_A(\text{Split-block design vs. split-plot design}) = \frac{MS_{sp}(E_A)}{MS_{sb}(E_A)} = 1. \tag{6}$$

Here ERE denotes the estimated relative efficiency.

For B-factor comparison, we have

$$ERE_B(\text{Split-block design vs. split-plot design}) = \frac{MS_{sp}(E_B)}{MS_{sb}(E_B)} < 1. \tag{7}$$

For A×B comparison we have

$$ERE_{A\times B}(\text{Split-block design vs. split-plot design}) = \frac{MS_{sp}(E_B)}{MS_{sb}(E_{AB})} > 1. \quad (8)$$

From (6), (7) and (8) we obtain: A is estimated with the same precision in both the split-plot and the split-block designs. B is estimated less precisely in the split-block design than in the split-plot design. Contrarily, $A \times B$ is estimated with higher precision in the split-block design than in the split-plot design. Hence, unless one is more interested in $A \times B$ comparison, the split-plot design seems more preferable.

4. Comparing efficiencies in the general case

The models will remain unchanged when we deal with the incomplete case for both the split-block and split-plot design. This is because in the incomplete case the structure of the designs and the randomization procedure will be the same as in the complete case. In the incomplete case, however, the mean treatment effects will no longer be the arithmetic mean because not every treatment occurs in every block. Instead, we can estimate an adjusted mean with the least-square method, thereby removing the effect of blocks. The degrees of freedom for the incomplete split-block design and the incomplete split-plot design are listed in Tables 4 and 5, respectively[1].

Table 4. The degrees of freedom for incomplete split-block design

Source	d.f.	MS
Replicates	$r-1$	$MS(R)$
A-factor	$s-1$	$MS(A)$
Error(A)	$ra-s-r+1$	$MS(E_A)$
B-factor	$t-1$	$MS(B)$
Error(B)	$rb-t-r+1$	$MS(E_B)$
A×B	$(s-1)(t-1)$	$MS(A\times B)$
Error(AB)	$rab-ra-rb+r-st+s+t-1$	$MS(E_{AB})$
Total	$rst-1$	

For a uniformity trial, the ANOVA is summarized in Table 6. We obtain our split-block design by adding a column structure to the superblocks of the split-plot design. Therefore the superblock and the row structure will remain unchanged. As a result, the precision of estimation for the row factor A in the split-block or that for

[1]For the incomplete cases in Table 4, 5, and 6, a is the whole–plot size in the split–plot design or the row size in the split–block design, b is the split–plot size in the split–plot design or the column size in the split–block design.

Table 5. The degrees of freedom for incomplete split-plot design

Source	d.f.	MS
Replicates	$r-1$	$MS(R)$
A-factor	$s-1$	$MS(A)$
Error(A)	$ra-s-r+1$	$MS(E_A)$
B-factor	$t-1$	$MS(B)$
A×B	$(s-1)(t-1)$	$MS(A\times B)$
Error(B)	$rab-ra-st+s$	$MS(E_B)$
Total	$rst-1$	

Table 6. ANOVA for uniformity trial in the incomplete case

(a) Incomplete split-block design

Sources	d.f.	MS
Replicate	$r-1$	
Error (A)	$r(a-1)$	$MS_{sb}(E_A)$
Error (B)	$r(b-1)$	$MS_{sb}(E_B)$
Error (AB)	$r(a-1)(b-1)$	$MS_{sb}(E_{AB})$
Total	$rab-1$	

(b) Incomplete split-plot design

Sources	d.f.	MS
Replicates	$r-1$	
Error (A)	$r(a-1)$	$MS_{sp}(E_A)$
Error (B)	$ra(b-1)$	$MS_{sp}(E_B)$
Total	$rab-1$	

the whole-plot factor A in the split-plot design will not be influenced, i.e.,

$$MS_{sb}(E_A) = MS_{sp}(E_A) \tag{9}$$

and

$$r(b-1)MS_{sb}(E_B) + r(a-1)(b-1)MS_{sb}(E_{AB}) = ra(b-1)MS_{sp}(E_B). \tag{10}$$

Hence,

$$\begin{aligned} MS_{sp}(E_B) &= \frac{r(b-1)MS_{sb}(E_B) + r(a-1)(b-1)MS_{sb}(E_{AB})}{ra(b-1)} \\ &= \frac{r(b-1)MS_{sb}(E_B) + r(a-1)(b-1)MS_{sb}(E_{AB})}{r(b-1) + r(a-1)(b-1)}. \end{aligned} \tag{11}$$

This means that $MS_{sp}(E_B)$ is a weighted average of $MS_{sb}(E_B)$ and $MS_{sb}(E_{AB})$.

Usually, $MS_{sb}(E_B)$ is greater than $MS_{sb}(E_{AB})$ (except for sampling errors) and $MS_{sp}(E_B)$ will be intermediate in size between $MS_{sb}(E_B)$ and $MS_{sb}(E_{AB})$.

Obviously, the same conclusion about the efficiency can be drawn for the incomplete case as in the complete case.

5. An example

In the example of incomplete split-block design with a potato variety (Hering and Mejza, 1997), the effect of irrigation and 10 pesticides on several variables was recorded. The ANOVA is summarized in Table 7. If we had used an incomplete split-plot design for the same data set, we would have get the ANOVA as in Table 8. It can be verified that:

– for comparing the effects of irrigation

$$ERE_A(\text{Split-block vs. split-plot}) = \frac{MS_{sp}(E_A)}{MS_{sb}(E_A)} = \frac{1.903}{1.903} = 1,$$

– for comparing the effects of pesticide, we have

$$ERE_B(\text{Split-block vs. split-plot}) = \frac{MS_{sp}(E_B)}{MS_{sb}(E_B)} = \frac{2.010}{2.896} = 0.694 < 1,$$

– for comparing the effects of irrigation × pesticide

$$ERE_{A\times B}(\text{Split-block vs. split-plot}) = \frac{MS_{sp}(E_B)}{MS_{sb}(E_{AB})} = \frac{2.010}{1.124} = 1.788 > 1.$$

We conclude that by using split-block design, the interaction between irrigation and pesticide can be estimated more precisely, leading to the less precise estimation of the effect of pesticide, while the precision of estimation for irrigation keeps unchanged.

Table 7. ANOVA of incomplete split-block design

Source	d.f.	*SS*	*MS*	*F* value	*P*-value
Superblock	14	356.368	25.455		
Irrigation	1	354.320	354.320	186.2242	0.0001
Superblock × irrigation	14	26.637	1.903		
Pesticide	9	2119.244	235.472	81.3076	0.0001
Superblock × pesticide	36	104.258	2.896		
Pesticide × irrigation	9	41.420	4.602	4.0937	0.0011
Residual	36	40.472	1.124		
Total	119	3042.720			

Table 8. ANOVA of incomplete split-plot design

Source	d.f.	SS	MS	F value	P-value
Superblock	14	356.368	25.455		
Irrigation	1	354.320	354.320	186.2242	0.0001
Superblock × irrigation	14	26.637	1.903		
Pesticide	9	2119.244	235.472	117.1415	0.0001
Pesticide × irrigation	9	41.420	4.602	2.2895	0.0254
Residual	72	144.731	2.010		
Total	119	3042.720			

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O efektywności układu split-block względem układu split-plot

STRESZCZENIE

Wprowadzenie dodatkowej struktury kolumnowej w układzie split-plot prowadzi do układu split-block w którym superbloki mają strukturę kratową. Artykuł bada względną efektywność tych dwu układów, również w przypadku niekompletnym.

SŁOWA KLUCZOWE: zagnieżdżona struktura wierszowo-kolumnowa, układy split-plot, układy split-block, względna efektywność.